Lecture 17: The Poisson distribution

Today's objectives

- Introduce the poisson distribution
- Calculate probabilities associated with a poisson
- Use R to calculate probabilites and cuttoff points associated with a poisson

When to use a Poisson?

Binomial vs. Poisson

- We have spent the last two lectures talking about how to handle a binary 0/1 outcome
- So far we've covered only the binomial distribution.
- Ch. 12 covers another distribution for counts: the Poisson distribution.
- In a binomial situation we are sampling observations/individuals and counting the number of "successes" in our sampled set.
- In a poisson situation we are sampling units of time or space and counting the number of events (which has no upper bound)
- The main distinction between Binomial and Poisson is that Poisson random variables have no upper bound, whereas the upper bound of a binomial random variable X was n, the size of the sample.

The Poisson distribution

- Most commonly, the Poisson distribution is used to model rare events. However this is not the only case where we would use Poisson.
- A Poisson distribution describes the count X of occurrences of a defined event in fixed, finite intervals of time or space when:

1. Occurrences are all independent (that is, knowing that one event has occurred does not change the probability that another event may occur- this was also true for our binomial distribution), and,
2. The probability of an occurrence is the same over all possible intervals or spacial units of the same size.

Examples of the Poisson distribution

Rare, but infectious diseases. For example, the number of deaths X attributed to typhoid fever on any given day over a long period of time, say 1 year, follows a Poisson distribution if:

a) The probability of a new death from typhoid fever in any one day is very small.
b) The number of cases reported in any two distinct periods of time are independent random variables.

citation: https://ani.stat.fsu.edu/~debdeep/p4_s14.pdf

Examples of the Poisson distribution

Rare events occurring on a surface area. The distribution of number of bacterial colonies growing on an agar plate. The number of bacterial colonies over the entire agar plate follows a Poisson distribution if:

a) The probability of finding any bacterial colonies in a small area is very small.
b) The events of finding bacterial colonies in any two areas are independent.

citation: https://ani.stat.fsu.edu/~debdeep/p4_s14.pdf

Poisson probabilities

Poisson probabilities

If X has the Poisson distribution with a mean number of occurrences per interval of μ, the possible values of X are 0, 1, 2, and so on. If k is any one of these values, then

$$P(X = k) = \frac{e^{-\mu}\mu^k}{k!}$$

- The above formula is the probability distribution function for a Poisson distribution.
- For example,

$$P(X = 2) = \frac{e^{-\mu}\mu^2}{2!}$$

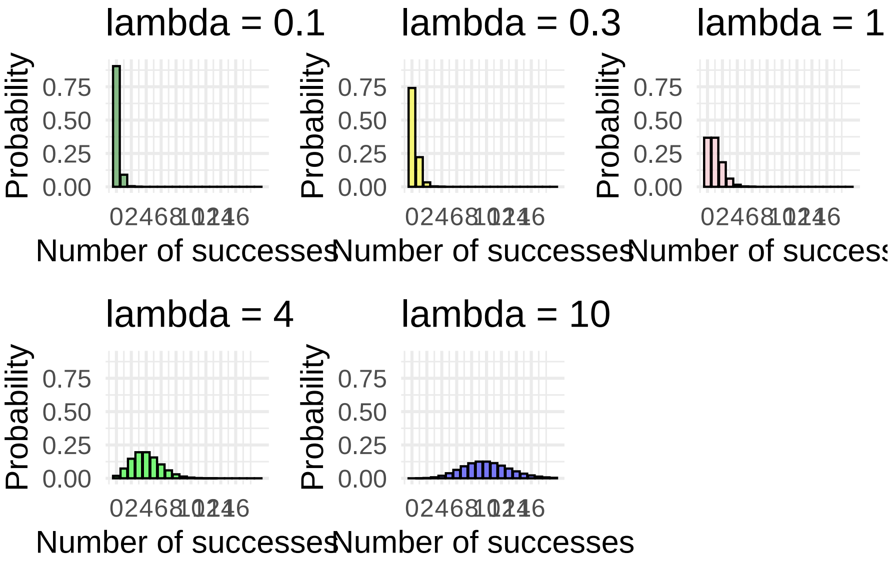
  will calculate the probability of observing two events for $X \sim Pois(\mu)$

How many parameters are there in a Poisson?

Mean and SD of a Poisson random variable

- The mean of a Poisson random variable is equal to μ.
- The variance is also equal to μ, and thus the SD is equal to $\sqrt{\mu}$.
- When the mean is large, so is the SD, and this makes for a flat and wide probability distribution.

Probability distribution of a Poisson random variable



R functions for the poisson

Poisson distribution in R

In R the poisson distribution is represented by `pois`

When working with the binomial distribution, we used `dbinom` and `pbinom` to calculate probabilities.

With the poisson distribution we will use `dpois` and `ppois`

In R, the function to calculate $P(X = x)$ for a poisson `dpois(x=?, lambda=?)`, where `lambda` (λ) is equal to the average μ (which is the book's notation).

More detail on syntax is *here*

Example: Mumps

In Iowa, the average monthly number of reported cases of mumps per year is 0.1. If we assume that cases of mumps are random and independent, the number X of monthly mumps cases in Iowa has approximately a Poisson distribution with $\mu = 0.1$. The probability that in a given month there is no more than 1 mumps case is:

$P(X \leq 1) = P(X = 0) + P(X = 1)$

$$P(X = k) = \frac{e^{-\mu}\mu^k}{k!}$$

Example: Mumps

$P(X \leq 1) = P(X = 0) + P(X = 1)$

$$P(X = k) = \frac{e^{-\mu}\mu^k}{k!}$$

$= \frac{e^{-0.1}0.1^0}{0!} + \frac{e^{-0.1}0.1^1}{1!}$ (note that $0! = 1$, by definition, and x^0=1, for any value of x.)

$= 0.9048 + 0.0905 = 0.9953$

Thus, we expect to only see 0 cases 90.5% of the months, and 1 case 9.05% of the time.

Example: Mumps calculated using R using ppois() and dpois()


```
dpois(x = 0, lambda = 0.1) + dpois(x = 1, lambda = 0.1)
```

```
## [1] 0.9953212
```

```
# or,

ppois(q = 1, lambda = 0.1) # notice that lambda is the parameter
```

```
## [1] 0.9953212
```

Example: Mumps, continued

Suppose you saw 4 cases of Mumps in a given month. What are the chances of seeing 4 or more cases in any given month?

```
1 - ppois(q = 3, lambda = 0.1) #careful, we used q = 3 here, why 3 and not 4?
```

```
## [1] 3.846834e-06
```

Could you have performed this calculation using `dpois()`?

If you saw 4 or more cases in any given month, this is very unlikely under this model. This suggests a substantial departure from the model, suggesting a contagious outbreak (no longer independent)

Example: Polydactyly

In the US, 1 in every 500 babies is born with an extra finger or toe. These events are random and independent. Suppose that the local hospital delivers an average of 268 babies per month. This means that for each month we expect to see 0.536 babies born with an extra finger or toe at that hospital (how do you calculate 0.536 here?). Let X be the count of babies born with an extra finger or toe in a month at that hospital.

a) What values can X take?
b) What distribution might X follow?
c) Give the mean and standard deviation of X.

Example: Polydactyly, continued

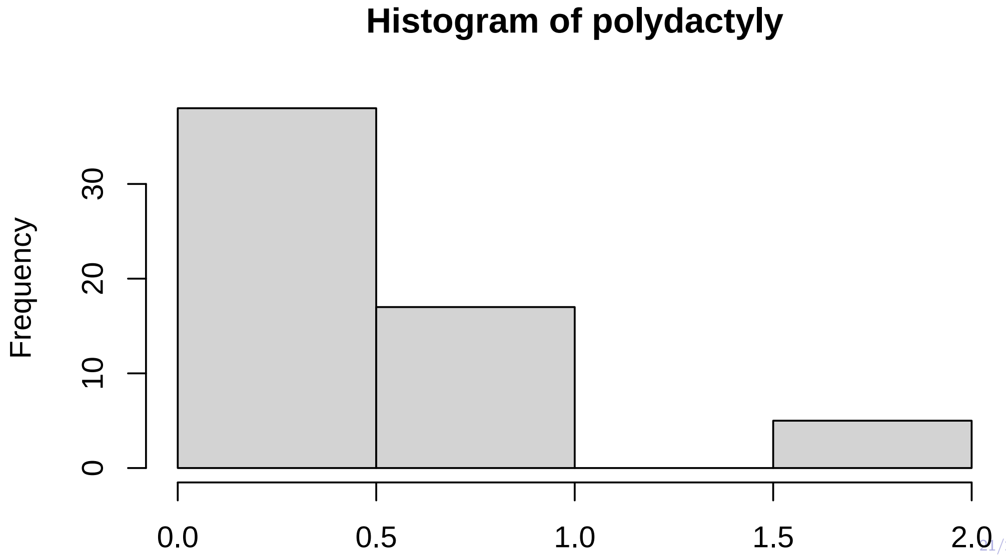
To get a sense of what the data might look like, use R to simulate data across five years (60 months) for this hospital.

```
set.seed(200)
polydactyly<-(rpois(n = 12*5, lambda = 0.536))
polydactyly
```

```
##  [1] 0 0 1 1 1 1 1 0 0 0 0 1 0 1 0 0 0 0 2 1 0 0 0 2 0 0 1 0 0 0 0 0 0 0 2
## [39] 0 0 1 1 1 0 2 1 0 0 1 1 0 0 1 0 0 0 0 0 0 1
```

Example: Polydactyly, continued

```
hist(polydactyly)
```



Example: Polydactyly, continued

```
mean(polydactyly)
```

```
## [1] 0.45
```

```
sd(polydactyly)
```

```
## [1] 0.6489888
```

More random number generation

Examining a stream of Poisson-distributed random numbers helps us get a sense of what these data look like. Can you think of a variable that might be Poisson-distributed according to one of these distributions?

```
rpois(100, lambda = 0.1)
```

```
##   [1] 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 2 0 0 0 0
##  [38] 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 2 0 0 0 0 0 0
##  [75] 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 1 0 1 1
```

```
rpois(100, lambda = 1)
```

```
##   [1] 1 2 0 2 0 1 0 0 0 1 2 0 0 0 1 1 0 2 2 0 1 0 2 1 1 1 0 0 0 1 2 0 4 0
##  [38] 0 1 0 1 1 1 0 2 1 0 2 0 1 0 0 0 0 1 1 2 0 1 2 0 0 1 1 0 2 0 0 0 0 1
##  [75] 0 1 1 1 0 2 0 2 1 1 2 2 1 2 2 0 0 1 2 3 1 1 0 0 2 1
```

check your understanding

For each of the following scenarios, what type of a distribution would be an appropriate model:

- The probability that 10 or more workers in a sample of 100 from a factory population test positive for SARS-COV-2 antibodies
- The probability that 10 or more bikes travel through an intersection near campus in an hour
- The glomerular filtration rate, a measure of kidney function

Comic Relief