The Normal Distribution

Corinne Riddell

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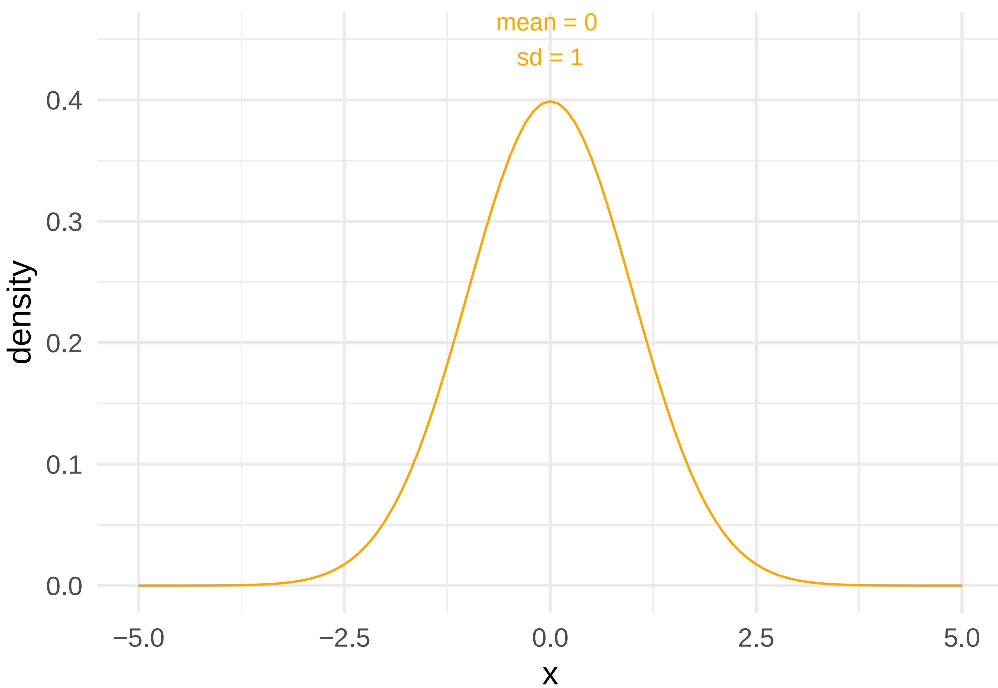
Learning objectives for today

- Learn about the Normal distribution centered at μ with a standard deviation of σ
- Learn about the standard Normal distribution where $\mu = 0$ and $\sigma = 1$ and compute z-scores
- Calculate cumulative probabilities below or above a given value for any specified Normal distribution using R
- Perform simple calculations by hand (using the 68-95-99.7 rule)

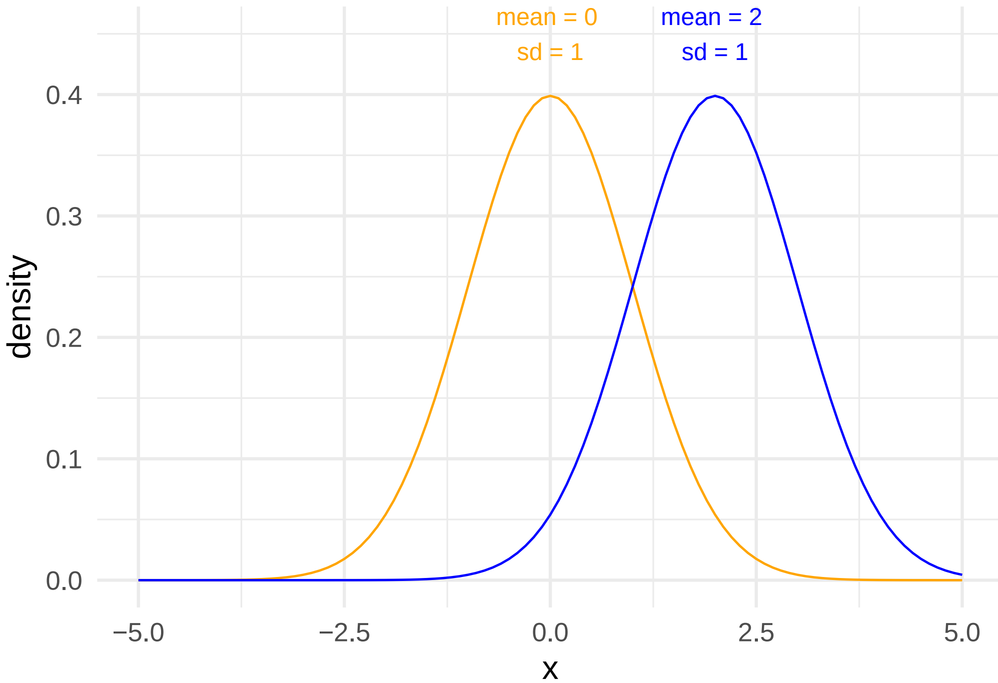
The Normal Distribution

- Here is the Normal distribution with mean of 0 (μ) and standard deviation of 1 (σ).
- It is:
  - symmetric
  - centered at μ



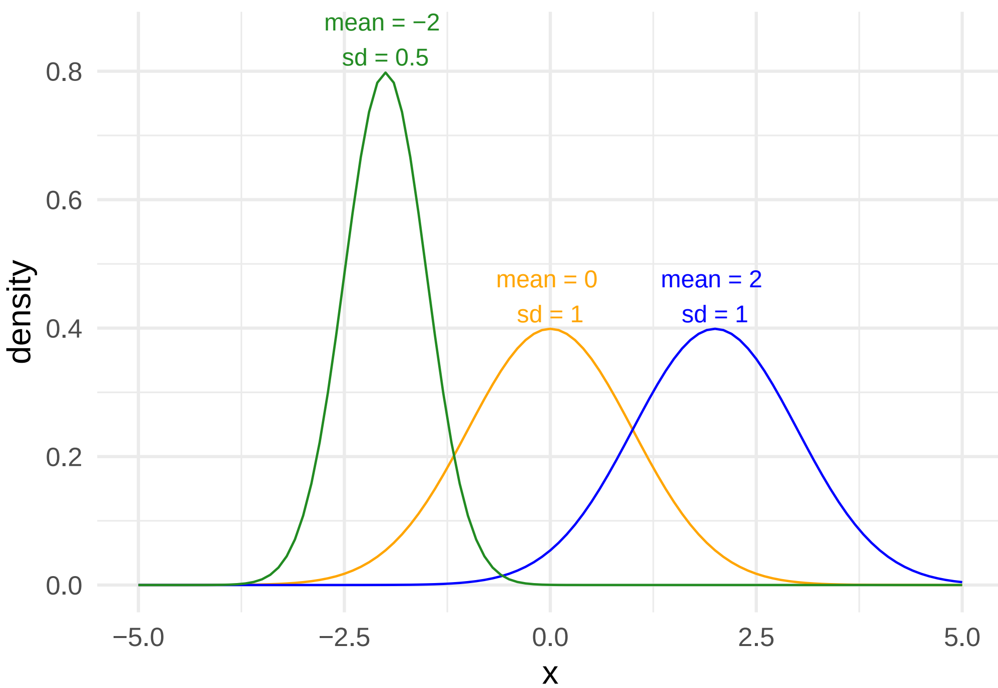
The Normal Distribution

- Let's add another Normal distribution, this one centered at 2, with the same standard deviation



The Normal Distribution

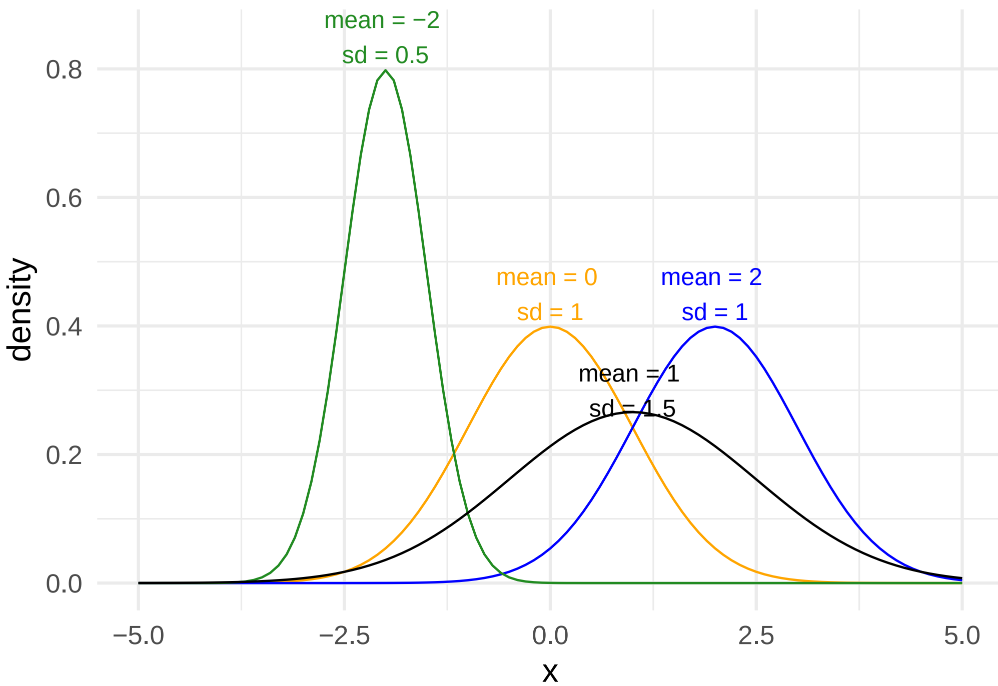
- Let's add a third Normal distribution, this one centered at -2, with a standard deviation of 0.5
- Notice how the distribution is narrowed (i.e., the spread is reduced)
- Why is the distribution "taller"?



The Normal Distribution

- Can you guess what a Normal distribution with $\mu = 1$ and $\sigma = 1.5$ would look like compared to the others?

The Normal Distribution



Properties of the Normal distribution

- the mean μ can be any value, positive or negative
- the standard deviation σ must be a positive number
- the mean is equal to the median (both $= \mu$)
- the standard deviation captures the spread of the distribution
- the area under the Normal distribution is equal to 1 (i.e., it is a density function)
- a Normal distribution is completely determined by its μ and σ

The 68-95-99.7 rule for all Normal distributions

- Approximately 68% of the data fall within one standard deviation of the mean
- Approximately 95% of the data fall within two standard deviations of the mean
- Approximately 99.7% of the data fall within three standard deviations of the mean

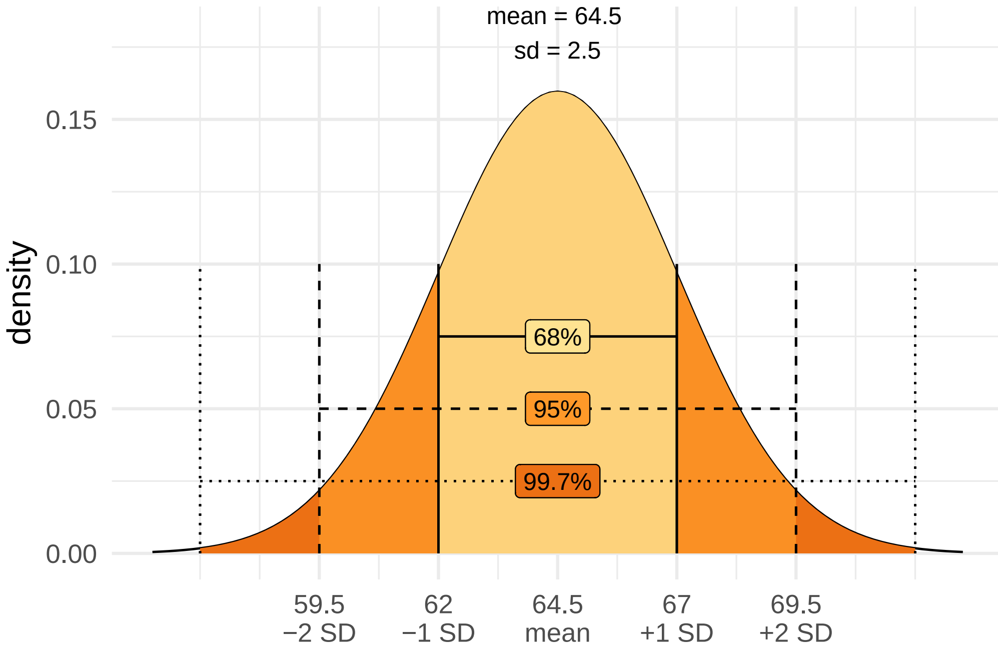
Written probabilistically:

- $P(\mu - \sigma < X < \mu + \sigma) \approx 68\%$
- $P(\mu - 2\sigma < X < \mu + 2\sigma) \approx 95\%$
- $P(\mu - 3\sigma < X < \mu + 3\sigma) \approx 99.7\%$

Calculations using the 68-95-99.7 rule

Example 11.1 from Baldi & Moore on the heights of young women. The distribution of heights of young women is approximately Normal, with mean $\mu = 64.5$ inches and standard deviation $\sigma = 2.5$ inches.

- We use notation to represent when a random variable follows a specific distribution. For example, letting H represent the random variable for the height of a young woman, we can then write $H \sim N(64.5, 2.5)$, to say that the random variable H follows a Normal distribution with a mean of 64.5 and a standard deviation of 2.5.

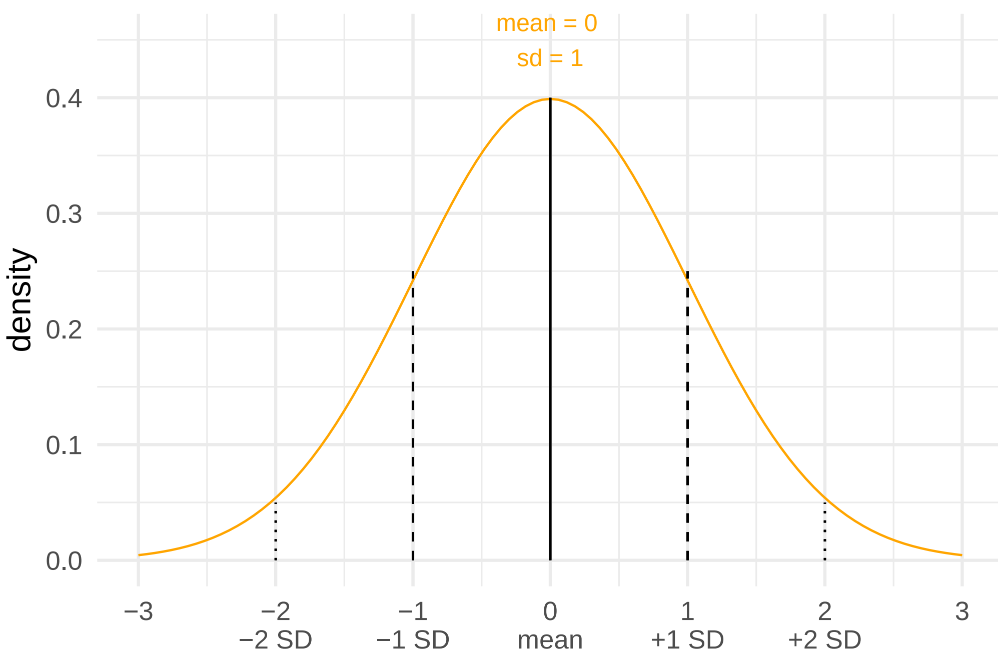


Calculations using the 68-95-99.7 rule

- What calculations could you do with these data alone?
- $P(62 < H < 67) = ?$
- $P(H > 62) = ?$

The standard Normal distribution

- The standard Normal distribution is the Normal distribution with $\mu = 0$ and $\sigma = 1$.
- We write: $N(0, 1)$ to denote this distribution
- $X \sim N(0, 1)$, implies that the random variable X is Normally distributed.



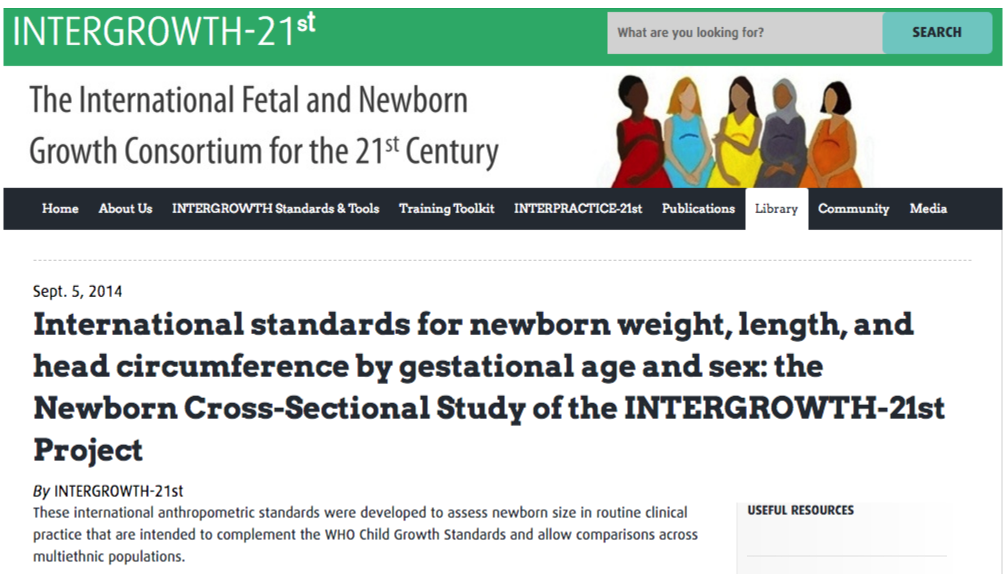
Standardizing Normally distributed data

- Any random variable that follows a Normal distribution can be standardized. This means we can transform its distribution from being centred at μ with a standard deviation of σ to another Normal distributuin with $\mu = 0$ and standard deviation of $\sigma = 1$
- If x is an observation from a distribution that has a mean μ and a standard deviation σ, the standardized value of x is calculated in the following way:

$$z = \frac{x - \mu}{\sigma}$$

- A standardized value is often called a **z-score**
- Interpretation: z is the number of standard deviations that x is above or below the mean of the data.
- We standardize values so that we can have this interpretation, which is agnostic to the underlying mean, standard deviation, and units of measure. Standardizing Normally-distributed data is a quick way to determine if a specific value is much higher or lower than the average value.

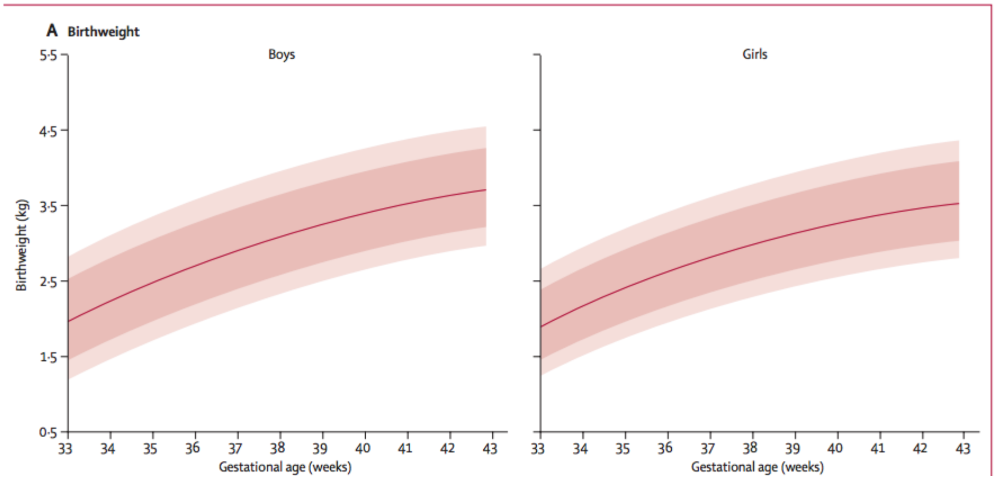
Standardizing Normally distributed data



Standardizing Normally distributed data

In this image, the solid red line shows the average birthweight as a function of gestational age for boys and girls.

What is the approximate average birthweight in kilograms for a boy delivered at 33 weeks?



Reference

Standardizing Normally distributed data

The International Newborn Standards

INTERGROWTH-21st

UNIVERSITY OF OXFORD

Birth weight (Boys)

Gestational age (weeks+days)	z scores						
	-3	-2	-1	0	1	2	3
33+0	0.63	1.13	1.55	1.95	2.39	2.88	3.47
33+1	0.67	1.17	1.59	1.99	2.43	2.92	3.51
33+2	0.71	1.21	1.63	2.03	2.47	2.96	3.55
33+3	0.75	1.25	1.67	2.07	2.50	2.99	3.59
33+4	0.79	1.29	1.71	2.11	2.54	3.03	3.62
33+5	0.83	1.33	1.75	2.15	2.58	3.07	3.66
33+6	0.87	1.37	1.79	2.18	2.62	3.11	3.70
34+0	0.91	1.40	1.82	2.22	2.65	3.14	3.73
34+1	0.95	1.44	1.86	2.26	2.69	3.18	3.77
34+2	0.98	1.48	1.90	2.29	2.73	3.21	3.80
34+3	1.02	1.51	1.93	2.33	2.76	3.25	3.84
34+4	1.05	1.55	1.97	2.36	2.80	3.28	3.87
34+5	1.09	1.58	2.00	2.40	2.83	3.32	3.91
34+6	1.12	1.62	2.04	2.43	2.86	3.35	3.94
35+0	1.16	1.65	2.07	2.47	2.90	3.38	3.97

- Birthweight z-scores for boys
- How does this relate to what you see on the previous slide?

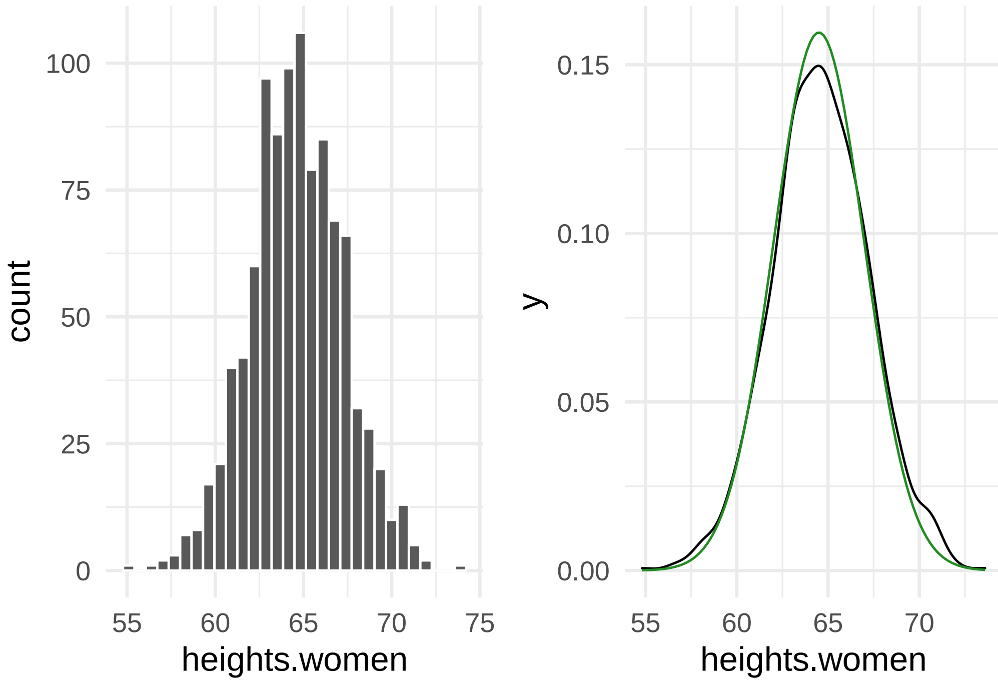
Simulating Normally distributed data in R

Suppose that we measured 1000 heights for young women:

```r
# students, rnorm() is important to know!
# this line of code generates 1000 rows of data from a Normal distribution with
# the specified mean and sd.
heights.women <- rnorm(n = 1000, mean = 64.5, sd = 2.5)
```

```
# this line of code puts this variable into a data frame
heights.women <- data.frame(heights.women)
```

We can plot the histogram of the heights, and see that they roughly follow from a Normal distribution. The green curve is a Normal distribution, and the black curve is the density plot based on the actual data:



Standardizing Normally distributed data in R

To standardize these data, we can apply the formula to compute the z-score:

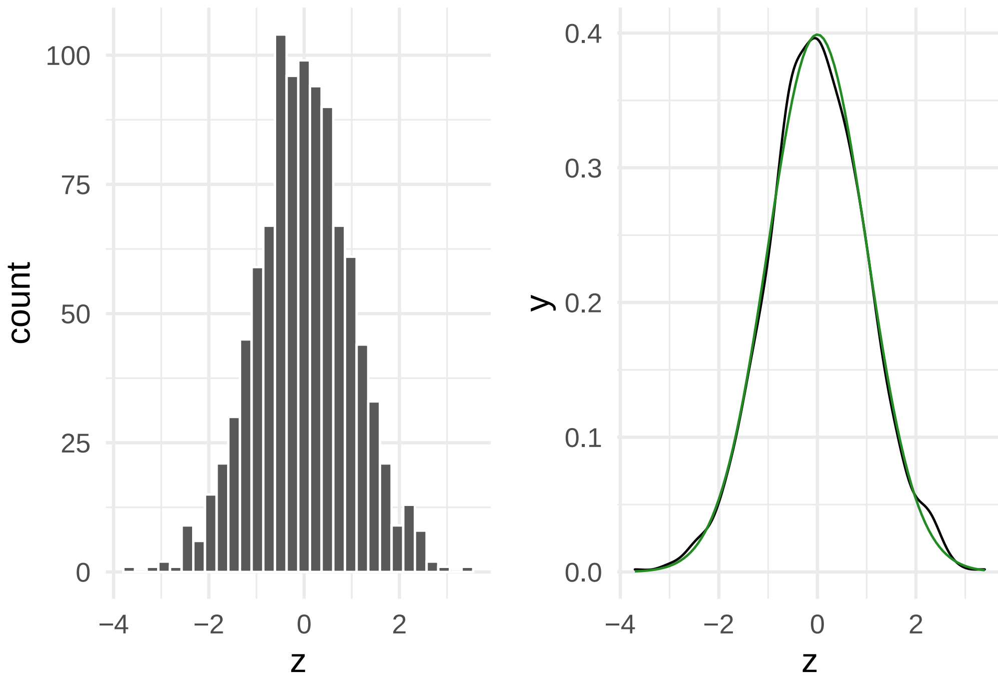
```
heights.women <- heights.women %>% mutate(mean = mean(heights.women),
                                          sd = sd(heights.women),
                                          z = (heights.women - mean)/sd)

head(heights.women)
```

```
##   heights.women     mean       sd            z
## 1      71.14762 64.61461 2.647221  2.467872178
## 2      66.51309 64.61461 2.647221  0.717158282
## 3      68.60105 64.61461 2.647221  1.505896876
## 4      64.60760 64.61461 2.647221 -0.002649075
## 5      66.88704 64.61461 2.647221  0.858420959
## 6      63.58915 64.61461 2.647221 -0.387372465
```

What would the distribution of the standardized heights look like?

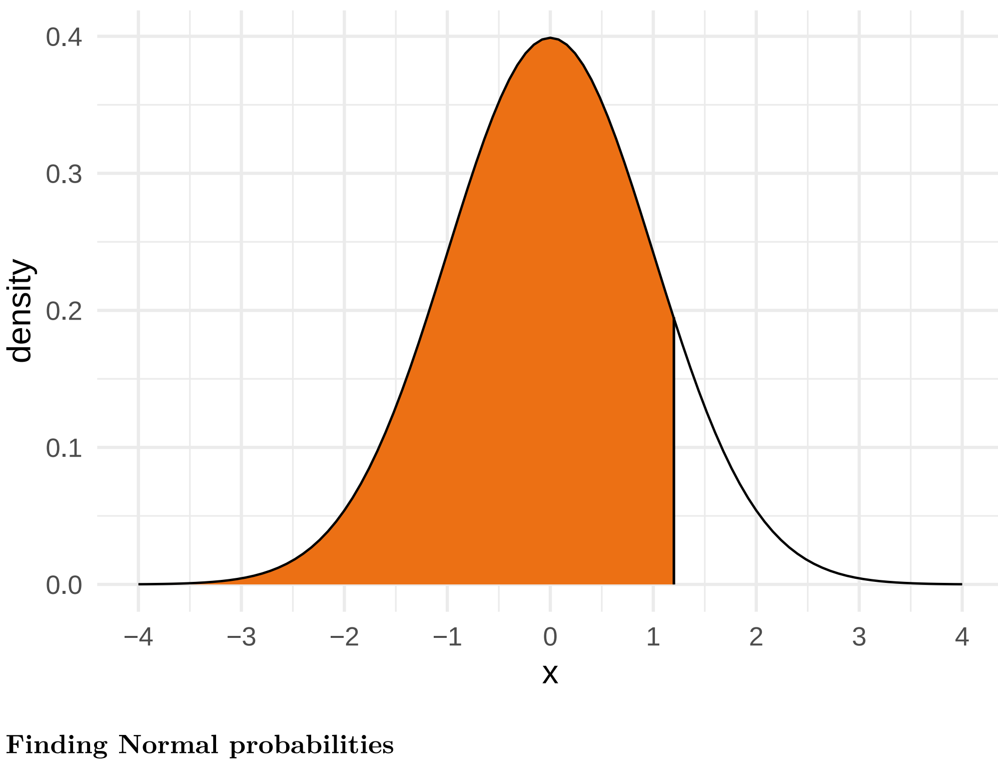
Standardizing Normally distributed data in R

How are these plots different from the previous ones? Hint: look at the x axis.



Finding Normal probabilities

- A **cumulative probability** for a value x in a distribution is the proportion of observations in the distribution that lie at or below x.
- Here is the cumulative probability for x=1.2



Finding Normal probabilities

- Recall that 100% of the sample space for the random variable X lies under its probability density function.
- What is the amount of the area that is below x = 1.2?

- To answer this question we use the `pnorm()` function. (Think: the **p** in `pnorm` stands for probability):

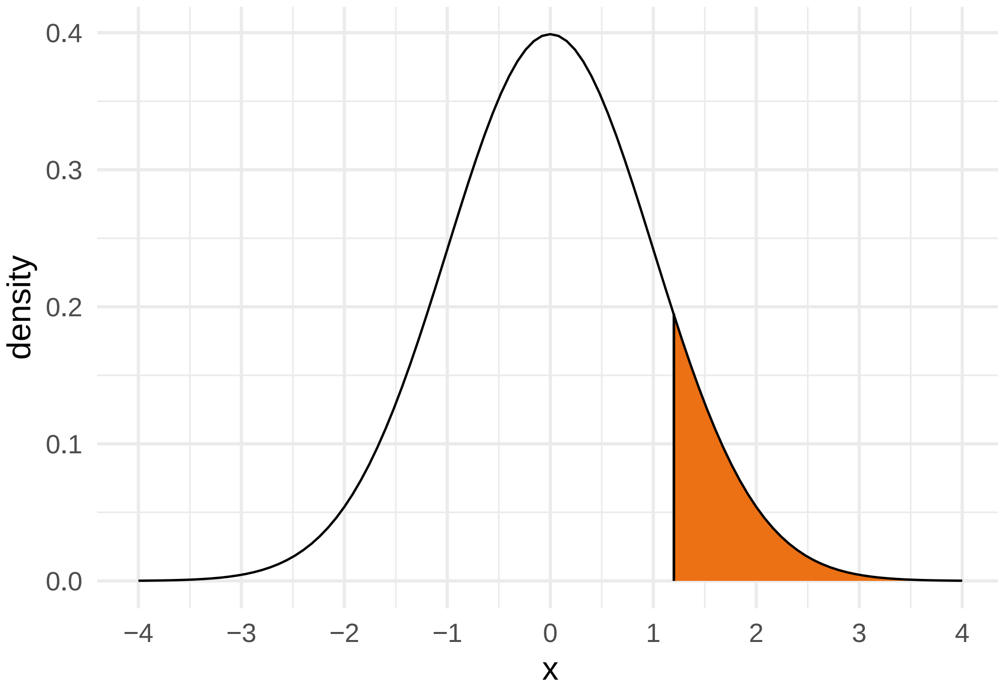
```
pnorm(q = 1.2, mean = 0, sd = 1)
```

```
## [1] 0.8849303
```

This says that approximately 88% of the probability lies below 1.2.

Finding Normal probabilities

What if we wanted the reverse: P(x>1.2)?




```
1 - pnorm(q = 1.2, mean = 0, sd = 1)
```

```
## [1] 0.1150697
```

Alternatively:

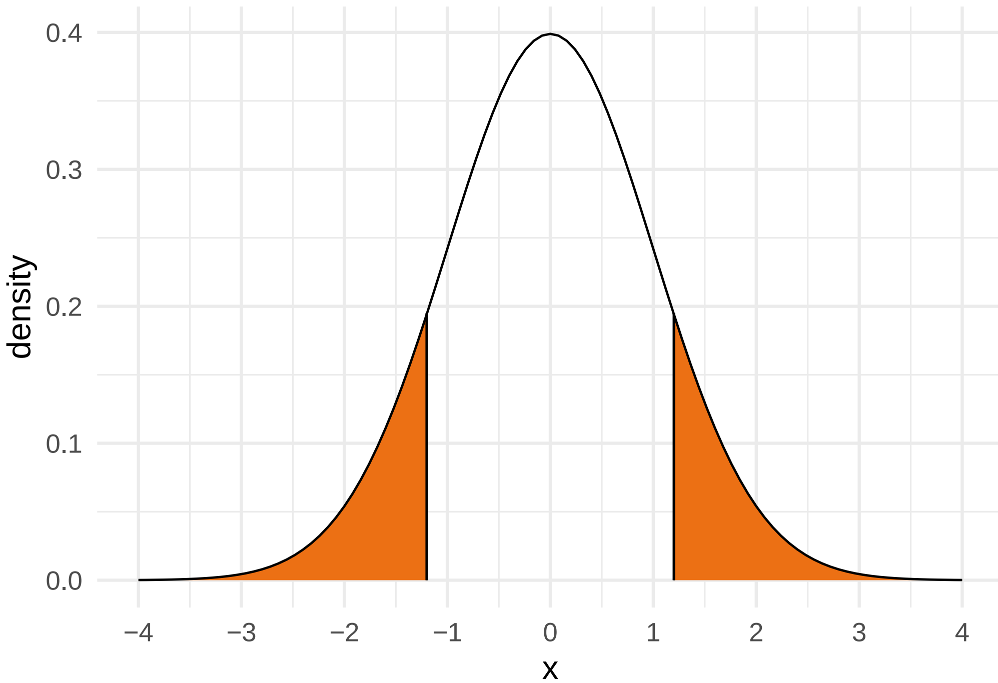
```
 pnorm(q = 1.2, mean = 0, sd = 1, lower.tail = F)
```

```
## [1] 0.1150697
```

So, 11.51% of the data is above x=1.2.

Finding Normal probabilities

What if we wanted two "tail" probabilities?: $P(x < -1.2 \text{ or } x > 1.2)$?



Finding Normal probabilities

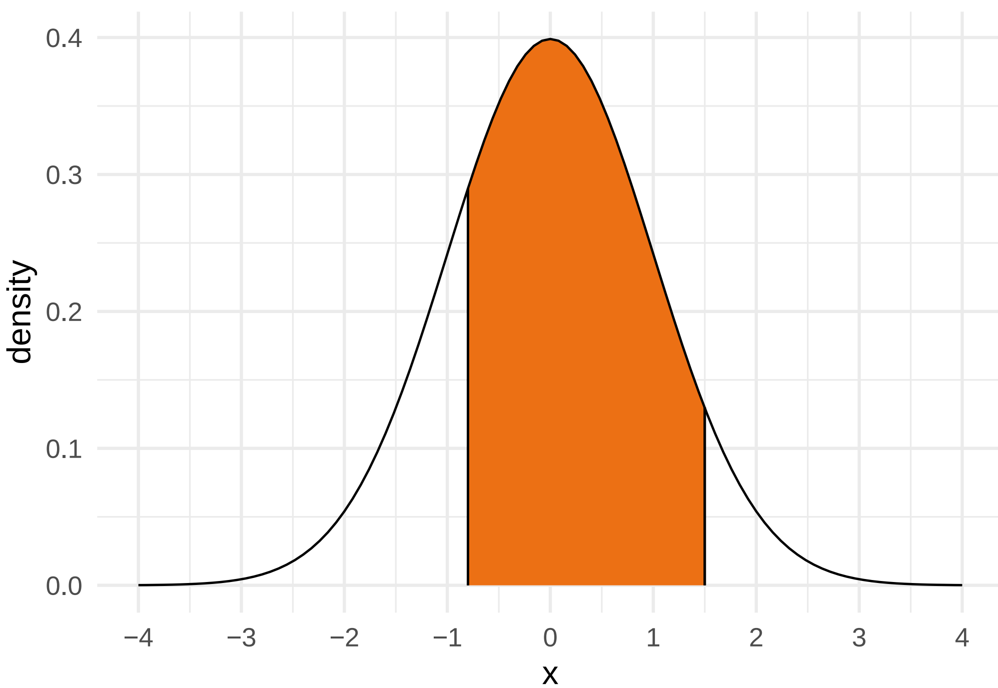
The trick: find one of the tails and then double the area because the distribution is symmetric:

```
pnorm(q = -1.2, mean = 0, sd = 1)*2
```

```
## [1] 0.2301393
```

Finding Normal probabilities

What if we wanted a range in the middle?: $P(-0.8 < x < 1.5)$?



Finding Normal probabilities

```
# step 1: calculate the probability *below* the upper bound (x=1.5)
pnorm(q = 1.5, mean = 0, sd = 1)
```

```
## [1] 0.9331928
```

```
# step 2: calculate the probability *below* the lower bound (x = -0.8)
pnorm(q = -0.8, mean = 0, sd = 1)
```

```
## [1] 0.2118554
```

```
# step 3: take the difference between these probabilities to get what's left in
# the middle
pnorm(q = 1.5, mean = 0, sd = 1) - pnorm(q = -0.8, mean = 0, sd = 1)
```

```
## [1] 0.7213374
```

Thus, 72.13% of the data is in the range -0.8 < x < 1.5.

Your turn

To diagnose osteoporosis, bone mineral density is measured. The WHO criterion for osteoporosis is a BMD score below -2.5. Women in their 70s have a much lower BMD than younger women. Their BMD ~ N(-2, 1). What proportion of these women have a BMD below the WHO cutoff?

Hint: you do not need to find a z-score!

```
#to fill in during class
```

Recap of functions used

- `rnorm(n = 100, mean = 2, sd = 0.4)`, to generate Normally distributed data from the specified distribution
- `pnorm(q = 1.2, mean = 0, sd = 2)`, to calculate the cumulative probability below a given value