Tree diagrams, absolute frequencies, and diagnostic testing

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Today's agenda

- Use absolute frequencies to calculate probabilities
- Use tree diagrams to calculate probabilities
- Apply these skills to diagnostic testing
  - Sensitivity, specificity, positive predictive value, negative predictive value, true positives, false positives, true negatives, and false negatives
- Learn Bayes' theorem

Unintended pregnancies

- Approximately 9% of all births in the US are to teen mothers (aged 15-19), 24% to younger adult mothers (ages 20-24) and the remaining 67% to older adult mothers (aged 25-44).
- A survey found that only 23% of births to teen mothers are intended. Among births to younger adult women, 50% are intended, and among older adult women 75% are intended

Define events using probability notation

Express all the percents on the previous slide using probability notation.

- Let M denote the age of the mother and B denote whether the birth was intended. Then we can define the events on the previous slides as:
  - $P(M = \text{teen}) = 0.09$
  - $P(M = \text{young adult}) = 0.24$
  - $P(M = \text{older adult}) = 0.67$
  - $P(B = \text{intended} | M = \text{teen}) = 0.23$
  - $P(B = \text{intended} | M = \text{young adult}) = 0.5$
  - $P(B = \text{intended} | M = \text{older adult}) = 0.75$

Question to answer

- What is the probability that any given live birth in the U.S. is unintended?
  - Rewrite this question as a probability statement
- We will review two ways to answer this question:
  a) Using absolute frequencies (not covered in the book)
  b) Using tree diagrams

Method a: Absolute Frequencies

- Pretend there are 1000 women. Given that 9%, 24%, and 67% of the mothers are teens, younger, and older mothers (respectively) this means that out of the 1000:
  - 90 are teens
  - 240 are younger mothers
  - 670 are older mothers

Method a: Absolute Frequencies

- Now, <u>conditional</u> on being a teen, 23% of the pregnancies are intended.
- This means that 90x23% = 20.7 teen mothers had intended pregnancies.
- We can calculate these joint probabilities for each age group:
  - 90 are teens, 90x23% = 20.7 teens with intended pregnancies (and 69.3 teens with unintended pregnancies).
  - 240 are younger mothers, 240 x50% = 120 younger mothers with intended pregnancies (and 120 younger mothers with unintended pregnancies).
  - 670 are older mothers, 670x75% = 502.5 older mothers with intended pregnancies (and 167.5 with unintended pregnancies).

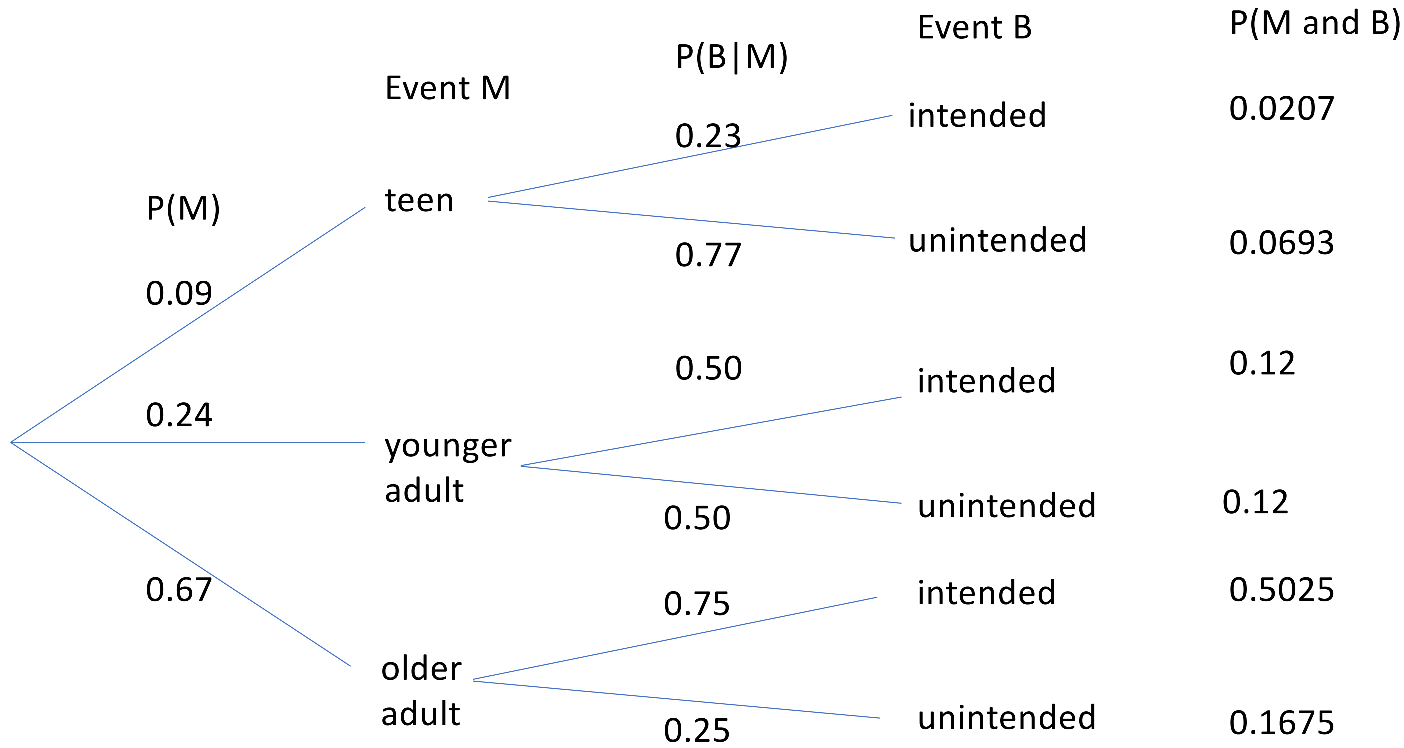
Method a: Absolute Frequencies

- Then, we can add on the number of unintended pregnancies across all the mothers:
  - 69.3 + 120 + 167.5 = 356.8
- The last step is to convert this back to a probability.
- To do that, remember that there were 1000 women in the population. So 356.8/1000 = 35.7%
- Conclusion: The chance that a live birth in the US is unintended is 35.7%.

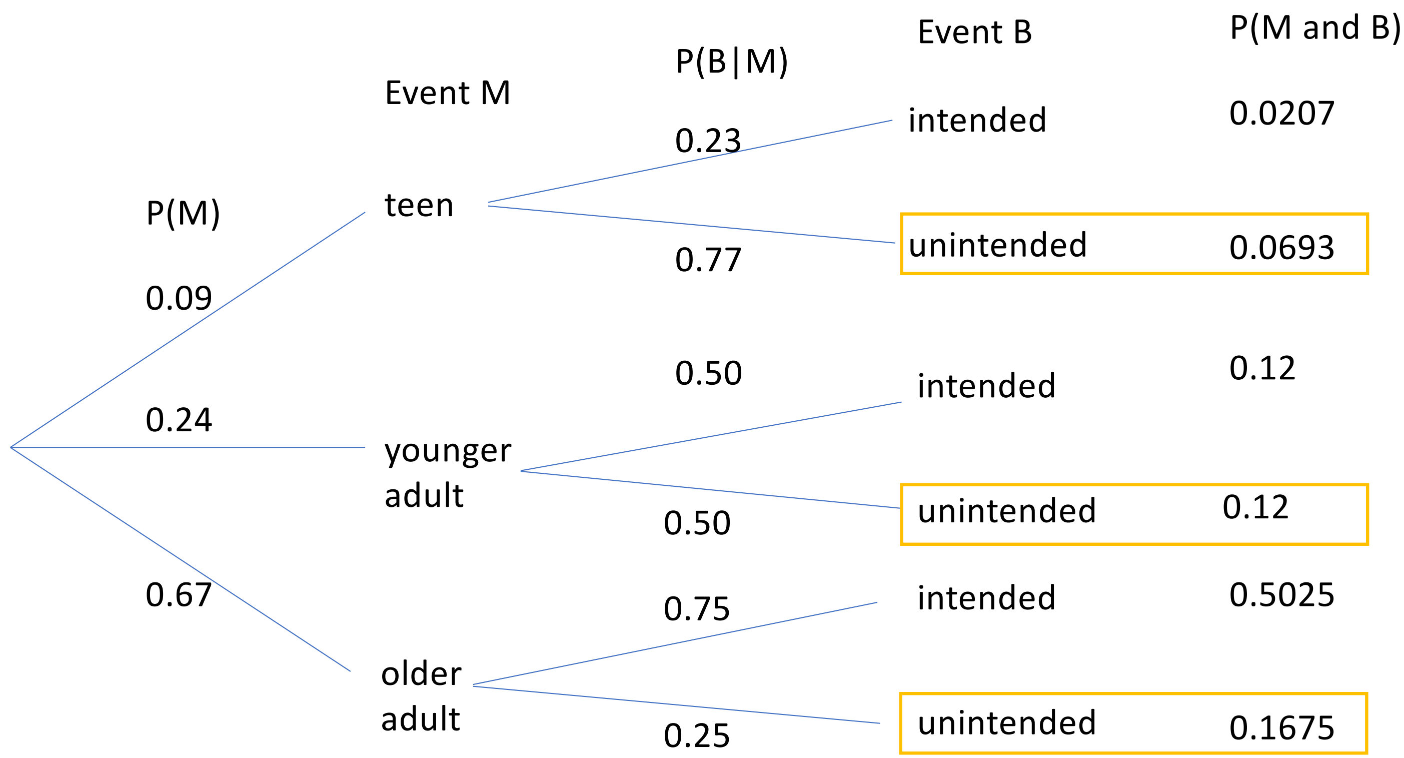
Method b: Tree diagram

- Rather than using absolute frequencies, you might prefer to draw this information using a tree diagram
- These diagrams are helpful when you know information about conditional probabilities and when the events of interest have more than two states (which is when Venn diagrams are used)

Method b: Tree diagram



Method b: Tree diagram



P(B=unintended) = 0.0693 + 0.12 + 0.1675 = 35.7%

Diagnostic Testing

Recall the question I asked a few days ago…

- Suppose that there is test for a specific type of cancer that has a 90% chance of testing positive for cancer if the individual truly has cancer and a 90% chance of testing negative for cancer when the individual does not have it.
- 1% of patients in the population have the cancer being tested for.
- What is the chance that a patient has cancer given that they test positive?
    - a) Between 0% - 24.9%
    - b) Between 25.0% - 49.9%
    - c) Between 50.0% - 74.9%
    - d) Between 75.0% - 100%

Rewrite this information using prob. notation

- Let C be the true cancer status. C = cancer for individuals who truly have cancer and C = no cancer for individuals who truly do not have cancer.
- Let T be the test result. T = positive for individuals who test positively for cancer and T = negative for individuals who test negative for cancer. Then:
  - P(C=cancer)=0.01
  - P(Test = positive|C=cancer) = 0.90
  - P(Test = negative|C=no cancer) = 0.90
- The question is "What is the chance that a patient has cancer given that they test positive". Rewrite the question using this probability notation.

Diagnostic testing definitions

- **Sensitivity**: The test's ability to appropriately give a positive result when a person tested has the disease, or **P(T = positive|C=cancer)**
- **Specificity**: The test's ability to appropriately give a negative result when a person tested does not have the disease, or **P(T = negative|C= no cancer)**

Diagnostic testing definitions

- **Positive predictive value**: The chance that a person truly has cancer, given that the test is positive, or **P(C=cancer|T=positive)**
- **Negative predictive value**: The chance that a person truly does not have cancer, given that the test is negative, or **P(C=no cancer|T=negative)**

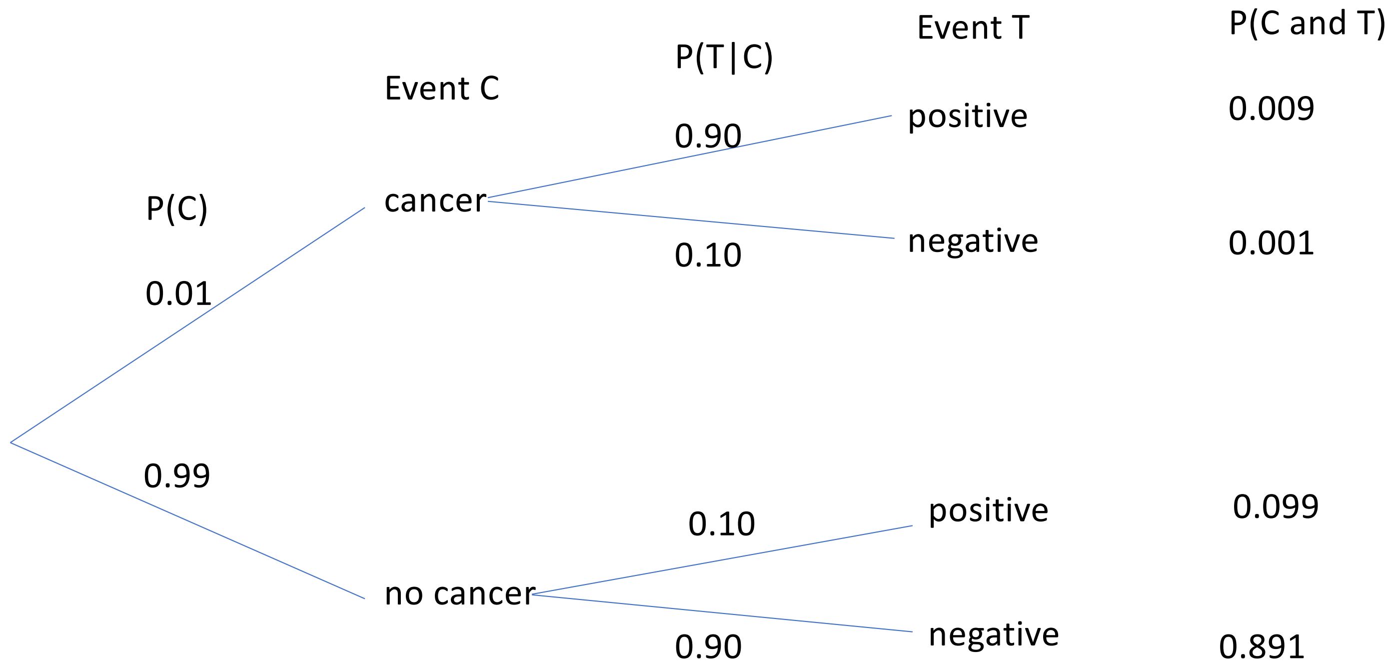
Back to the question

- Going back to the question… The question provided us information on the test's **sensitivity** and **specificity** as well as the **prevalence** of cancer in the underlying population
- The question asks us for the test's **positive predictive value**.
- We can use absolute frequencies or a tree diagram to answer the question.

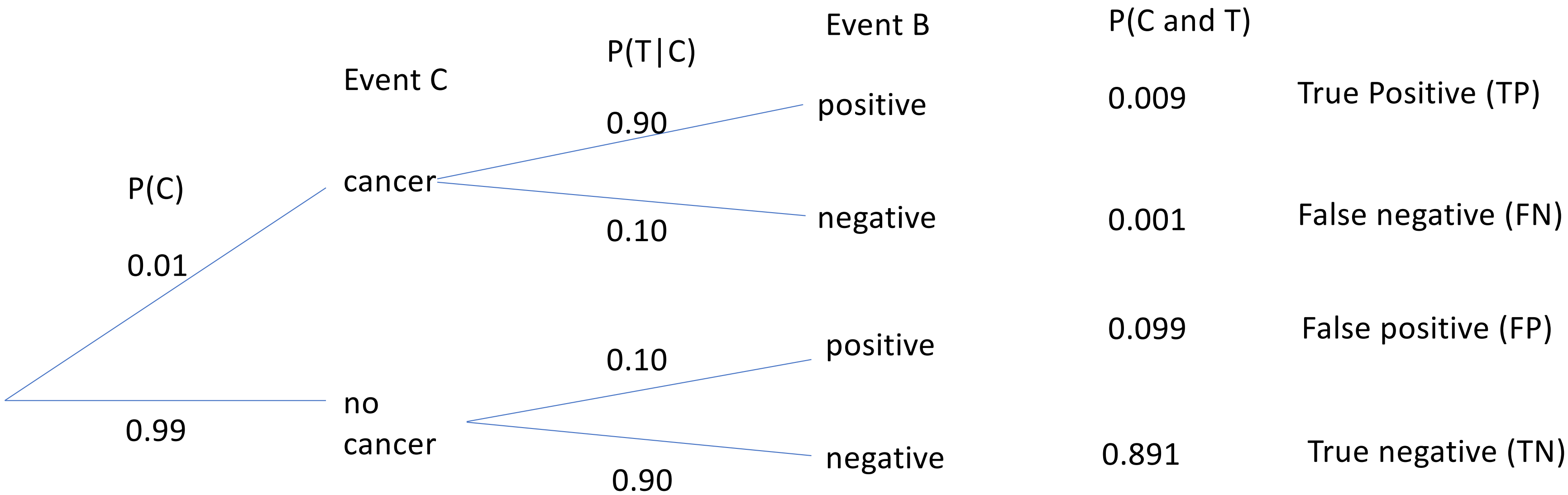
Absolute frequency approach

- Suppose that there are 1000 women in the population
- Translate the probabilities provided into absolute frequencies:
  - 1% truly have cancer → 10 women truly have cancer, 990 women do not.
  - 90% sensitivity → Among the 10 who truly have cancer, 9 women will test positive and 1 will test negative.
  - 90% specificity → Among the 990 who do not have cancer, 891 will test negative, and 99 will test positive.
  - So, we have 9 + 99 = 108 women detected with cancer
  - Of these 108 women, only 9 truly have cancer. Thus, 9/108 = 8.3% of those detected for cancer actually have it.

Method b: Tree diagram



Method b: Tree diagram



P(C=cancer|T=positive) = P(cancer & test positive)/P(test positive)
= P(cancer & test positive)/[P(test positive & cancer) + P(test positive & no cancer)]
= P(true positive)/[P(true positive) + P(false positive)]
= 0.009/(0.009 + 0.099) = 8.3%

Bayes’ Theorem

- To answer this question, we started with information on P(T|C) and P(C) and used it to calculate P(C|T).
- We can generalize how we did this using a rule known as Bayes’ Theorem.
- To begin, recall the formula for conditional probability from last class:

$$P(A|B) = \frac{P(A\&B)}{P(B)}$$

Bayes' Theorem

- To begin, recall the formula for conditional probability from last class:

$$P(A|B) = \frac{P(A\&B)}{P(B)} \text{[Formula 1]}$$

- This formula also implies:

$$P(B|A) = \frac{P(A\&B)}{P(A)}$$

which can be rearranged as: $P(B|A) \times P(A) = P(A\&B)$ [Formula 2]

Bayes’ Theorem

- Plug Formula 2 into Formula 1:

$$P(A|B) = \frac{P(B|A)\times P(A)}{P(B)} \text{[Formula 3]}$$

- If A only has two states, either A occurs or it does not (A’ occurs), then P(B) can be partitioned into two pieces:

$$P(B) = P(B\&A) + P(B\&A') = P(B|A)P(A) + P(B|A')P(A')$$

- Then we can plug in this result into Formula 3:

$$P(A|B) = \frac{P(B|A)\times P(A)}{P(B|A)P(A) + P(B|A')P(A')}$$

Bayes' Theorem

$$P(A|B) = \frac{P(B|A) \times P(A)}{P(B|A)P(A) + P(B|A')P(A')}$$

- This is Bayes' Theorem
- It allows to calculate a conditional probability (here, P(A|B)), when we only have information on the reverse condition (P(B|A)), as well as information on the overall probability of A (P(A))
- This is how we calculated the positive predictive value, P(C=cancer|T=+), when we only knew the Sensitivity (P(T=+|C=cancer)), Specificity (P(T=-|C=no cancer)), and Prevalence of cancer (P(C=cancer))

Bayes’ Theorem, Generalized

- Rather than only having A and A', suppose that A could take the values 1, 2, 3, and so on through A=k, where each of these states are disjoint and there probabilities are non-zero and add to 1.
- Then for B whose probability is not 0 or 1,

$$P(A_i|B) = \frac{P(B|A_i) \times P(A_i)}{P(B|A_1) \times P(A_1) + P(B|A_2) \times P(A_2) + \ \dots + P(B|A_k) \times P(A_k)}$$

- Don’t worry too much about understanding this formula
- Rather, focus on practicing the calculations for diagnostic testing like the one shown on the previous slide.
- You can watch this video (6 mins) to see how Bayes’ Theorem is using in AI today.

Recap

- Absolute frequencies or tree diagrams
  - Use the method you like best to solve for probabilities
  - Or, use a Venn diagram. Apply the method that makes the most sense to you and suits the question.
- Diagnostic testing
  - Key lesson: Just because sensitivity and specificity are high, this does not imply that the positive predictive value is also high. In lab, you will explore why this is the case
- Bayes’ Theorem
  - We used it without event knowing it!
  - Don’t worry about the formula, just know how to solve for probabilities using the method that you understand best.